Results when Sampling from the Normal Distribution

Linghai Liu

December 25, 2022

Contents

1 Properties of a Random Sample

1.1 Independence

Theorem 1.1. Suppose $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$
Define random variables $\bar{X}$ and S^2 as the following:

$$\bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Then $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$

Proof. $\forall t > 0$, the characteristic function of $\bar{X}$ is:

$$\begin{aligned}
\phi_{\bar{X}}(t) &= \phi_{\sum_{i=1}^{n} X_i}\left(\frac{t}{n}\right) \\
&= \prod_{i=1}^{n} \phi_{X_i}\left(\frac{t}{n}\right) \\
&= \left(\phi_{X_i}\left(\frac{t}{n}\right)\right)^n \\
&= \exp\{i\frac{t}{n}\mu n - \frac{\sigma^2(\frac{t}{n})^2}{2}n\} \\
&= \exp\{it\mu - \frac{\frac{\sigma^2}{n}t^2}{2}\}
\end{aligned}$$

which is the characteristic function of $\mathcal{N}(\mu, \frac{\sigma^2}{n})$ □

To facilitate with the following proofs, we introduce the following theorems (Theorem 4.6.11 and Theorem 4.6.12) in *Statistical Inference (2001)* by Casella & Berger [1], which are the generalizations of two lemmas and are not proved in the textbook.

Theorem 1.2. Let $X_1, \ldots, X_n$ be random vectors. Then they are mutually independent random vectors if and only if there exist functions $g_i(x_i), i = 1, \ldots, n$ such that the joint pdf or pmf of $(X_1, \ldots, X_n)$ can be written as

$$f(x_1, \ldots, x_n) = g_1(x_1) \cdot \cdots \cdot g_n(x_n)$$

Proof. i. ($\Longrightarrow$) By the definition of independence,

$$f(x_1, \dots, x_n) = f_1(x_1) \cdot \dots \cdot f_n(x_n)$$

where $f_i(x_i)$ is the marginal probability density functions of X_i. So we have found these functions.

ii. ($\Longleftarrow$) Denote d_i as the dimension of each random vector and let $d := \sum_{i=1}^{n} d_i$. Define

$$C_i := \int_{\mathbb{R}^{d_i}} g_i(x_i) dx_i$$

Since $f(x_1, \dots, x_n)$ is the joint pdf, then

$$\begin{aligned}
1 &= \int_{\mathbb{R}^d} f(x_1, \dots, x_n) dx_1 \dots dx_n \quad \text{definition of pdf} \\
&= \int_{\mathbb{R}^{\Sigma_{i=1}^{d} d_i}} g_1(x_1) \cdot \dots \cdot g_n(x_n) dx_1 \dots dx_n \quad \text{by assumption} \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}^{d_i}} g_i(x_i) dx_i \quad \text{by Fubini's Theorem in Euclidean space} \\
&= \prod_{i=1}^{n} C_i
\end{aligned}$$

Furthermore, the marginal distribution of $X_i, i = 1, \dots, n$ can be given by

$$f_i(x_i) := g_i(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} C_j \tag{1.1}$$

which could be easily verified. Note that

$$\prod_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} C_j = \prod_{i=1}^{n} \prod_{j=1}^{n} C_j \Big/ \prod_{i=1}^{n} \prod_{\substack{j=1 \\ j = i}}^{n} C_j = \prod_{i=1}^{n} 1 \Big/ \prod_{i=1}^{n} C_i = 1 \tag{1.2}$$

And using this,

$$\begin{aligned}
f(x_1, \dots, x_n) &= \prod_{i=1}^{n} g_i(x_i) \quad \text{by assumption} \\
&= \left(\prod_{i=1}^{n} g_i(x_i) \right) \left(\prod_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} C_j \right) \quad \text{by Equation 1.2} \\
&= \prod_{i=1}^{n} \left(g_i \prod_{\substack{j=1 \\ j \neq i}}^{n} C_j \right) \\
&= \prod_{i=1}^{n} f_i(x_i) \quad \text{by Equation 1.1}
\end{aligned}$$

Since $f_i(x_i)$ is the marginal distribution of $X_i, i = 1, \dots, n$, $(X_1, \dots, X_n)$ are independent random vectors by definition of independence.

□

Theorem 1.3. Let $X_1, \dots, X_n$ be independent random vectors. Let $g_i(x_i)$ be a function only of x_i whose range is a subset of $\mathbb{R}$, $i = 1, \dots, n$. Then the random variables $U_i := g_i(X_i), i = 1, \dots, n$, are mutually independent.

Proof. Denote d_i as the dimension of each random vector and let $d := \sum_{i=1}^{n} d_i$.
$\forall u_i \in \mathbb{R}, i = 1, \ldots, n$, define

$$A_{u_i}^{(i)} := \{x \in \mathbb{R}^{d_i} : g_i(x) \leq u_i\}$$

The joint cumulative distribution function (cdf) of $g_1(X_1), \ldots, g_n(X_n)$ is:

$$\begin{aligned} F(u_1, \ldots, u_n) &= \mathbb{P}\{g_1(X_1) \leq u_1, \ldots, g_n(X_n) \leq u_n\} \\ &= \mathbb{P}\{X_1 \in A_{u_1}^{(1)}, \ldots, X_n \in A_{u_n}^{(n)}\} \\ &= \prod_{i=1}^{n} \mathbb{P}\{X_i \in A_{u_i}^{(i)}\} \quad \text{by independence of } X_i\text{'s} \end{aligned}$$

Denote X_{ij} as the j-th entry of the i-th random vector X_i, where $1 \leq i \leq n, 1 \leq j \leq d_i$.
The joint pdf of $g_1(X_1), \ldots, g_n(X_n)$ is:

$$\begin{aligned} f(u_1, \ldots, u_n) &= \frac{\partial^d}{\prod_{i=1}^{n} \prod_{j=1}^{d_i} \partial x_{ij}} F(u_1, \ldots, u_n) \\ &= \frac{\partial^{\sum_{i=1}^{n} d_i}}{\prod_{i=1}^{n} \prod_{j=1}^{d_i} \partial x_{ij}} \prod_{k=1}^{n} \mathbb{P}\{X_k \in A_{u_k}^{(k)}\} \\ &= \prod_{i=1}^{n} \frac{\partial^{d_i}}{\prod_{j=1}^{d_i} \partial x_{ij}} \mathbb{P}\{X_i \in A_{u_i}^{(i)}\} \\ &= \prod_{i=1}^{n} \left(\prod_{j=1}^{d_i} \frac{\partial}{\partial x_{ij}} \right) \mathbb{P}\{g_i(X_i) \leq u_i\} \end{aligned}$$

Hence, the joint pdf is the product of a series of n functions where the i-th function is of $g_i(X_i)$ only, for each i.
By Theorem 1.2, we conclude that $g_1(X_1), \ldots, g_n(X_n)$ are independent. □

Theorem 1.4. Let $\bar{X}$ and S^2 defined as in Theorem 1.1. Then $\bar{X}$ and S^2 are independent.

Proof.

$$\begin{aligned} S^2 &= \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \\ &= \frac{1}{n-1} \left[\sum_{i=2}^{n} (X_i - \bar{X})^2 + (X_1 - \bar{X})^2 \right] \\ &= \frac{1}{n-1} \left[\sum_{i=2}^{n} (X_i - \bar{X})^2 + \left(\sum_{i=2}^{n} (X_i - \bar{X}) \right)^2 \right] \end{aligned}$$

because $\sum_{i=1}^{n} (X_i - \bar{X}) = 0$.
The joint probability density function of $X_1, X_2, \ldots, X_n$ is:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(x_i - \mu)^2}{2\sigma^2}\} \quad \text{by independence} \\ &= \frac{1}{(2\pi)^{n/2} \sigma^n} \exp\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\} \end{aligned}$$

We would like to perform a change of variables on the probability density function with the following:

$$Y_1 = \bar{X}, Y_2 = X_2 - \bar{X}, Y_3 = X_3 - \bar{X}, \ldots, Y_n = X_n - \bar{X}$$

The realized values of Y_i's and X_i's relate as follows:

$$y_1 = \bar{x}, y_2 = x_2 - \bar{x}, y_3 = x_3 - \bar{x}, \ldots, y_n = x_n - \bar{x}$$

Solving these n equations, we obtain:

$$x_1 = y_1 - \sum_{i=1}^{n} y_n, x_2 = y_2 + y_1, x_3 = y_3 + y_1, \ldots, x_n = y_n + y_1$$

The Jacobian J of the transformation is:

$$J = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{bmatrix}$$

The determinant of J is

$$\det J = \begin{vmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{vmatrix} = \frac{1}{n} \quad \text{expanding over the first column}$$

The second row is obtained by adding the first row of J to all following rows. This is valid because of the property that the determinant does not change by elementary row operations.

Then the joint probability density function of $Y_1, Y_2, \ldots, Y_n$ is:

$$f(y_1, y_2, \ldots, y_n) = (\det J^{-1}) \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{1}{2\sigma^2} \left((y_1 - \sum_{i=2}^{n} y_i - \mu)^2 + \sum_{i=2}^{n} (y_i + y_1 - \mu)^2 \right)\} \tag{1.3}$$

Calculating the terms in large parenthesis:

$$\begin{aligned} & (y_1 - \sum_{i=2}^{n} y_i - \mu)^2 + \sum_{i=2}^{n} (y_i + y_1 - \mu)^2 \\ &= y_1^2 + (\sum_{i=2}^{n} y_i)^2 + \mu^2 - 2y_1 \sum_{i=2}^{n} y_i - 2y_1\mu + 2\mu \sum_{i=2}^{n} y_i \\ &+ \sum_{i=2}^{n} y_i^2 + (n-1)y_1^2 + (n-1)\mu^2 + 2y_1 \sum_{i=2}^{n} y_i - 2\mu \sum_{i=2}^{n} y_i - 2(n-1)y_1\mu \\ &= n\mu^2 - 2ny_1\mu + ny_1^2 + \sum_{i=2}^{n} y_i^2 + (\sum_{i=2}^{n} y_i)^2 \end{aligned}$$

Substituting back to Equation 1.3:

$$\begin{aligned}
f(y_1, y_2, \dots, y_n) &= \frac{1}{\det J} \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{1}{2\sigma^2}(n\mu^2 - 2ny_1\mu + ny_1^2)\} \exp\{-\frac{1}{2\sigma^2}\left(\sum_{i=2}^n y_i^2 + (\sum_{i=2}^n y_i)^2\right)\} \\
&= \frac{n}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{n}{2\sigma^2}(y_1 - \mu)^2\} \exp\{-\frac{1}{2\sigma^2}\left(\sum_{i=2}^n y_i^2 + (\sum_{i=2}^n y_i)^2\right)\} \\
&= \frac{n}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{n}{2\sigma^2}(\bar{x} - \mu)^2\} \exp\{-\frac{1}{2\sigma^2}\left(\sum_{i=2}^n (x_i - \bar{x})^2 + (\sum_{i=2}^n (x_i - \bar{x}))^2\right)\} \quad \text{change to } x_i\text{'s} \\
&= \frac{n}{(2\pi)^{n/2}\sigma^n} \exp\{-\frac{n}{2\sigma^2}(\bar{x} - \mu)^2\} \exp\{-\frac{s^2}{2\sigma^2}\} \quad \text{by definition of } s^2 \\
&:= C \cdot g_1(y_1) g_2(y_2, \dots, y_n) := \tilde{g}_1(y_1) g_2(y_2, \dots, y_n)
\end{aligned}$$

Hence, $Y_1 = \bar{X}$ and $Y_2, \dots, Y_n$ are independent by Theorem 1.2. As S^2 is a function only of $Y_2, \dots, Y_n$, by Theorem 1.3 we conclude that $\bar{X}$ and S^2 are independent. □

1.2 The Chi-squared distribution

Definition 1.5 (χ^2 distribution)**.** For a χ^2 distribution with p degrees of freedom, the probability density function is

$$f(x) = \frac{1}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}} x^{\frac{p}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x>0}$$

This is actually a Gamma distribution with shape $\frac{p}{2}$ and scale 2.

Lemma 1.6. If $W_1 \sim \chi^2_{p_1}$ and $W_2 \sim \chi^2_{p_2}$ that are independent, then $W_1 + W_2 \sim \chi^2_{p_1+p_2}$

Proof. The characteristic function of Gamma(k, θ) is

$$(1 - it\theta)^{-k}, \forall t > 0$$

Hence, $\forall t > 0$, by property of χ^2 distribution listed in Definition 1.5,

$$\phi_{W_1}(t) = (1 - 2ti)^{-\frac{p_1}{2}}, \; \phi_{W_2}(t) = (1 - 2ti)^{-\frac{p_2}{2}}$$

The characteristic function of $W_1 + W_2$ is then

$$\phi_{W_1+W_2}(t) = \phi_{W_1}(t)\phi_{W_2}(t) = (1 - 2ti)^{-\frac{p_1+p_2}{2}}$$

which is the characteristic function of $\chi^2_{p_1+p_2}$. □

Lemma 1.7. If $Z \sim \mathcal{N}(0, 1)$, then $Z^2 \sim \chi^2_1$

Proof. $\forall z \geq 0$, the probability distribution function of Z^2 is:

$$\begin{aligned}
F_{Z^2}(z) &= \mathbb{P}(Z^2 \leq z) \\
&= \mathbb{P}(-\sqrt{z} \leq Z \leq \sqrt{z}) \\
&= \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt
\end{aligned}$$

This is differentiable by the Fundamental Theorem of Calculus, so

$$\begin{aligned}
f_{Z^2}(z) = \frac{d}{dz} F_{Z^2}(z) &= \frac{d}{dz} \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{2\sqrt{z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z}{2}} - \frac{-1}{2\sqrt{z}} e^{-\frac{z}{2}} \\
&= \frac{1}{\sqrt{2\pi z}} e^{-\frac{z}{2}}
\end{aligned}$$

The probability density function of χ_1^2 is

$$f(x) = \frac{1}{\Gamma(\frac{1}{2})2^{\frac{1}{2}}} x^{\frac{1}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x>0} = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}} \mathbb{1}_{x>0}$$

using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Hence, $Z^2 \sim \chi_1^2$ □

Theorem 1.8. Let S^2 is the same as defined in Theorem 1.1. Then $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$

Proof. For $1 \leq k \leq n$, define:

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i, \; S_k^2 = \frac{1}{k-1}\sum_{i=1}^{k}(X_i - \bar{X}_k)^2$$

Claim: for $k \geq 2$,

$$\frac{(k-1)S_k^2}{\sigma^2} = \frac{(k-2)S_{k-1}^2}{\sigma^2} + \frac{k-1}{k\sigma^2}(X_k - \bar{X}_{k-1})^2$$

The claim is proved as the following direct calculation:

$$\begin{aligned}
\frac{(k-1)S_k^2}{\sigma^2} - \frac{(k-2)S_{k-1}^2}{\sigma^2} &= \frac{1}{\sigma^2}\left[\sum_{i=1}^{k}(X_i - \bar{X}_k)^2 - \sum_{i=1}^{k-1}(X_i - \bar{X}_{k-1})^2\right] \\
&= \frac{1}{\sigma^2}\left[\sum_{i=1}^{k}(X_i^2 - 2\bar{X}_k X_i + \bar{X}_k^2) - \sum_{i=1}^{k-1}(X_i^2 - 2\bar{X}_{k-1}X_i + \bar{X}_{k-1}^2)\right] \\
&= \frac{1}{\sigma^2}\left[X_k^2 - k\bar{X}_k^2 + (k-1)\bar{X}_{k-1}^2\right] \quad \text{by definition of } \bar{X}_k \text{ and } \bar{X}_{k-1} \\
&= \frac{1}{\sigma^2}\left[X_k^2 - k(\frac{(k-1)\bar{X}_{k-1} + X_k}{k})^2 + (k-1)\bar{X}_{k-1}^2\right] \\
&= \frac{1}{\sigma^2}\left[\frac{k-1}{k}X_k - \frac{2(k-1)}{k}X_k\bar{X}_{k-1} + \frac{k-1}{k}\bar{X}_{k-1}^2\right] \\
&= \frac{k-1}{k\sigma^2}(X_k - \bar{X}_{k-1})^2
\end{aligned}$$

With this claim, we can prove the theorem by an argument of induction.

i. Base case: when $n = 2$,

$$\begin{aligned}
\frac{(n-1)S^2}{\sigma^2} = \frac{S_2^2}{\sigma^2} &= \frac{1}{2\sigma^2}(X_2 - X_1)^2 \quad \text{by the claim} \\
&= \left(\frac{X_2 - X_1}{\sqrt{2}\sigma}\right)^2
\end{aligned}$$

By an argument similar to Theorem 1.1 using characteristic functions, $\left(\frac{X_2 - X_1}{\sqrt{2}\sigma}\right)^2 \sim \mathcal{N}(0,1)$.

By Lemma 1.7, $\frac{(n-1)S^2}{\sigma^2} = \left(\frac{X_2 - X_1}{\sqrt{2}\sigma}\right)^2 \sim \chi_1^2$

Hence, base case holds.

ii. Assume that the theorem holds for $n = k \geq 2$, i.e.,

$$\frac{(k-1)S^2}{\sigma^2} \sim \chi_{k-1}^2$$

iii. Inductive step: when $n = k+1$, by the claim above,

$$\frac{kS_{k+1}^2}{\sigma^2} = \frac{(k-1)S_k^2}{\sigma^2} + \frac{k}{(k+1)\sigma^2}(X_{k+1} - \bar{X}_k)^2$$

By the assumption in step (ii), the first term on the right hand side has a χ^2_{k-1} distribution. By Lemma 1.6, to prove that the theorem holds for $n = k+1$, we only need to prove that

$$\frac{k}{(k+1)\sigma^2}(X_{k+1} - \bar{X}_k)^2 \sim \chi^2_1$$

By the setup and the conclusion in Theorem 1.1, we know that $X_{k+1} \sim \mathcal{N}(\mu, \sigma^2)$ and $\bar{X}_k \sim \mathcal{N}(\mu, \frac{\sigma^2}{k})$. Consider the random variable $X_{k+1} - \bar{X}_k$. The expectation is 0 because of the linearity of expectation. Since this random variable is a linear combination of normal random variables, it is still normally distributed. Its variance is:

$$\begin{aligned}\operatorname{Var}(X_{k+1} - \bar{X}_k) &= \operatorname{Var}(X_{k+1}) + (-1)^2 \operatorname{Var}(\bar{X}_k) \quad \text{since } X_{k+1} \text{ and } X_1, X_2, \ldots, X_k \text{ are independent} \\ &= \sigma^2 + \frac{\sigma^2}{k} = \frac{(k+1)\sigma^2}{k}\end{aligned}$$

Normalizing $X_{k+1} - \bar{X}_k$, we obtain

$$\frac{X_{k+1} - \bar{X}_k}{\sqrt{\frac{k+1}{k}}\sigma} \sim \mathcal{N}(0,1)$$

Then

$$\frac{k}{(k+1)\sigma^2}(X_{k+1} - \bar{X}_k)^2 = \left(\frac{X_{k+1} - \bar{X}_k}{\sqrt{\frac{k+1}{k}}\sigma}\right)^2 \sim \chi^2_1$$

by Lemma 1.7, as the term inside the parentheses has standard normal distribution.

Therefore, the theorem holds by induction. □

1.3 Student's t distribution and Snedecor's F distribution

In the following, we introduce the t-distribution and the F-distribution, as well as their respective properties and connections between the two families of distributions.

Given n observations from the same normal distribution, we can use Theorem 1.8 to build a confidence interval for the true mean μ of the normal distribution provided its variance σ^2. However, usually we lack this information as well. One solution is to replace the standard deviation σ by the square root of the sample variance S^2. And this leads us to Student's t-distribution.

Following Theorem 1.1,

$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \iff \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim \mathcal{N}(0,1)$$

by standardizing the normal random variable $\bar{X}$. Then

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \cdot \frac{\sigma}{S} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \cdot \frac{1}{\sqrt{\frac{S^2}{\sigma^2}}}$$

By Theorem 1.8, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Combining this with Theorem 1.4, we can write $\frac{\sqrt{n}(\bar{X} - \mu)}{S}$ as:

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} = Z \cdot \frac{1}{\sqrt{\frac{V}{n-1}}}$$

where $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi^2_{n-1}$ are independent.

Definition 1.9 (t-distribution)**.** Let $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi_p^2$ be independent ($p \geq 2$ and $p \in \mathbb{N}^*$). Then

$$T := \frac{Z}{\sqrt{\frac{V}{p}}}$$

has a Student's t-distribution with p degrees of freedom, denoted as $T \sim t_p$.

To step further, suppose we have two samples we can define Snedecor's F-distribution:

$$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_X, \sigma_X^2),\ \bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i,\ S_X^2 := \frac{1}{n-1}\sum_{i=1}^{n-1}(X_i - \bar{X})^2$$

$$Y_1, \ldots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_Y, \sigma_Y^2),\ \bar{Y} := \frac{1}{n}\sum_{i=1}^{n} Y_i,\ S_Y^2 := \frac{1}{n-1}\sum_{i=1}^{n-1}(Y_i - \bar{Y})^2$$

where all X_i's and Y_j's are independent.

Definition 1.10 (F-distribution)**.** The quotient

$$F := \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}$$

is defined to admit an F-distribution with $n-1$ and $m-1$ degrees of freedom, denoted as $F \sim F_{n-1,m-1}$

Equivalently, by Theorem 1.8,

$$\frac{(n-1)S_X^2}{\sigma_X^2} \sim \chi_{n-1}^2,\ \frac{(m-1)S_Y^2}{\sigma_Y^2} \sim \chi_{m-1}^2$$

and these two random variables are independent. Using this, let $U \sim \chi_{n-1}^2$ and $V \sim \chi_{m-1}^2$ be independent,

$$\frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} = \frac{U/(n-1)}{V/(m-1)} \sim F_{n-1,m-1}$$

Theorem 1.11 (Properties of F-distribution)**.** Let F be a random variable with an F-distribution with p and q degrees of freedom ($p, q \geq 2$ and $p, q \in \mathbb{N}^*$). Then

i. The probability density function of F is

$$f_F(x) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{x^{\frac{p}{2}-1}}{\left(\frac{p}{q}x+1\right)^{\frac{p+q}{2}}}\mathbb{1}_{x>0}, \quad \forall x \in \mathbb{R}$$

ii. When $q > 2$, then the expectation of F is $\frac{q}{q-2}$.

iii. When $q > 4$, then the variance of F is $\frac{2q^2(p+q-2)}{p(q-4)(q-2)^2}$.

Proof. i. Let $U \sim \chi_p^2$ and $V \sim \chi_q^2$ be independent. F can be expressed as $\frac{U/p}{V/q}$ by Definition 1.10.

The joint density of (U,V) is

$$\begin{aligned}
f_{UV}(u,v) &= f_U(u)f_V(v) \quad \text{by independence} \\
&= \frac{1}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}}u^{\frac{p}{2}-1}e^{-\frac{u}{2}}\mathbb{1}_{u>0}\frac{1}{\Gamma(\frac{q}{2})2^{\frac{q}{2}}}v^{\frac{q}{2}-1}e^{-\frac{v}{2}}\mathbb{1}_{v>0} \\
&= \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})2^{\frac{p+q}{2}}}u^{\frac{p}{2}-1}v^{\frac{q}{2}-1}e^{-\frac{u+v}{2}}\mathbb{1}_{u>0}\mathbb{1}_{v>0} \quad \forall u, v \in \mathbb{R}
\end{aligned}$$

Let $W := V$. We are going to change from variables (U, V) to (F, W). The Jacobian is

$$J = \begin{bmatrix} \frac{\partial F}{\partial U} & \frac{\partial F}{\partial V} \\ \frac{\partial W}{\partial U} & \frac{\partial W}{\partial V} \end{bmatrix} = \begin{bmatrix} \frac{q}{pV} & \frac{\partial F}{\partial V} \\ 0 & 1 \end{bmatrix} \implies \det J = \frac{q}{pV}$$

For the realized values $F = x$ and $W = w$,

$$u = \frac{pxw}{q}, v = w$$

Then the joint pdf of F and W is

$$\begin{aligned} f_{FW}(x, w) &= (\det J^{-1}) f_{UV}\left(\frac{pxw}{q}, w\right) \\ &= \frac{pw}{q} \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})2^{\frac{p+q}{2}}} \left(\frac{pxw}{q}\right)^{\frac{p}{2}-1} w^{\frac{q}{2}-1} e^{-\frac{\frac{pxw}{q}+w}{2}} \mathbb{1}_{x>0}\mathbb{1}_{w>0} \\ &= \frac{1}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{\frac{p}{2}} w^{\frac{p+q}{2}-1} e^{-\frac{1}{2}(\frac{p}{q}x+1)w} x^{\frac{p}{2}-1} \mathbb{1}_{x>0}\mathbb{1}_{w>0} \end{aligned}$$

Recall that for a Gamma distribution with shape k and scale θ, the pdf is

$$\frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}} := g(x; k, \theta)$$

and

$$\int_0^\infty g(x; k, \theta) dx = 1$$

Hence we have the following:

$$\int_0^\infty x^{k-1} e^{-\frac{x}{\theta}} = \Gamma(k)\theta^k$$

The density of F, $f_F(x)$, is just the marginal density, which can be calculated as follows:

$$\begin{aligned} f_F(x) &= \int_{\mathbb{R}} f_{FW}(x, w) dw \\ &= \frac{1}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{\frac{p}{2}} x^{\frac{p}{2}-1} \mathbb{1}_{x>0} \int_0^\infty w^{\frac{p+q}{2}-1} e^{-\frac{1}{2}(\frac{p}{q}x+1)w} dw \\ &= \frac{1}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{\frac{p}{2}} x^{\frac{p}{2}-1} \mathbb{1}_{x>0} \Gamma\left(\frac{p+q}{2}\right) \left(\frac{2}{\frac{p}{q}x+1}\right)^{\frac{p+q}{2}} \qquad \text{let } k = \frac{p+q}{2}, \theta = \frac{2}{\frac{p}{q}x+1} \\ &= f_F(x) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{x^{\frac{p}{2}-1}}{\left(\frac{p}{q}x+1\right)^{\frac{p+q}{2}}} \mathbb{1}_{x>0} \end{aligned}$$

ii. The expectation of F is

$$\mathbb{E}[F] = \int_0^\infty x \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{x^{\frac{p}{2}-1}}{\left(\frac{p}{q}x+1\right)^{\frac{p+q}{2}}} dx$$

Perform a change of variables:

$$\frac{p}{q}x = \frac{p+2}{q-2}y, \quad dx = \frac{q(p+2)}{p(q-2)}dy$$

$$\begin{aligned}
\mathbb{E}[F] &= \int_0^\infty \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{x^{\frac{p}{2}}}{\left(\frac{p}{q}x+1\right)^{\frac{p+q}{2}}}dx \\
&= \frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{q-2}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\int_0^\infty \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{q-2}{2}\right)} \frac{\left(\frac{p+2}{q-2}y\right)^{\frac{p+2}{2}-1}}{\left(1+\frac{p+2}{q-2}y\right)^{\frac{p+q}{2}}}\frac{q(p+2)}{p(q-2)}dy \\
&= \frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{q-2}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\frac{q}{p}\int_0^\infty \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{q-2}{2}\right)}\left(\frac{p+2}{q-2}\right)^{\frac{p+2}{2}} \frac{y^{\frac{p+2}{2}-1}}{\left(1+\frac{p+2}{q-2}y\right)^{\frac{p+q}{2}}}dy \\
&= \frac{\Gamma(\frac{p+2}{2})\Gamma(\frac{q-2}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\frac{q}{p}\cdot 1 \quad \text{the integrand is pdf of } Y \sim F_{p+2,q-2} \text{ by i} \\
&= \frac{\frac{p}{2}\Gamma(\frac{p}{2})\Gamma(\frac{q-2}{2})q}{\Gamma(\frac{p}{2})\frac{q-2}{2}\Gamma(\frac{q-2}{2})p} \quad \text{by property of the } \Gamma \text{ function} \\
&= \frac{q}{q-2}
\end{aligned}$$

iii. To find the variance of F, we use the identity

$$\text{Var}[F] = \mathbb{E}[F^2] - (\mathbb{E}[F])^2$$

$$\mathbb{E}[F^2] = \int_0^\infty x^2 \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\left(\frac{p}{q}\right)^{\frac{p+4}{2}-1}\frac{q}{p}\frac{x^{\frac{p}{2}-1}}{\left(\frac{p}{q}x+1\right)^{\frac{p+q}{2}}}dx$$

Perform a change of variables:

$$\frac{p}{q}x = \frac{p+4}{q-4}w, \quad dx = \frac{q(p+4)}{p(q-4)}dw$$

$$\begin{aligned}
\mathbb{E}[F^2] &= \frac{\Gamma(\frac{p+4}{2})\Gamma(\frac{q-4}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\int_0^\infty \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p+4}{2}\right)\Gamma\left(\frac{q-4}{2}\right)} \frac{\left(\frac{p+4}{q-4}w\right)^{\frac{p+4}{2}-1}}{\left(1+\frac{p+4}{q-4}w\right)^{\frac{p+q}{2}}}\frac{q(p+4)}{p(q-4)}dw \\
&= \frac{\Gamma(\frac{p+4}{2})\Gamma(\frac{q-4}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\frac{q^2}{p^2}\int_0^\infty \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p+4}{2}\right)\Gamma\left(\frac{q-4}{2}\right)}\left(\frac{p+4}{q-4}\right)^{\frac{p+4}{2}} \frac{w^{\frac{p+4}{2}-1}}{\left(1+\frac{p+4}{q-4}w\right)^{\frac{p+q}{2}}}dw \\
&= \frac{\Gamma(\frac{p+4}{2})\Gamma(\frac{q-4}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\frac{q^2}{p^2}\cdot 1 \quad \text{the integrand is pdf of } W \sim F_{p+4,q-4} \text{ by i} \\
&= \frac{q^2\Gamma(\frac{p}{2})\frac{p}{2}\frac{p+2}{2}\Gamma(\frac{q-4}{2})}{p^2\Gamma(\frac{p}{2})\frac{q-2}{2}\frac{q-4}{2}\Gamma(\frac{q-4}{2})} \quad \text{by property of the } \Gamma \text{ function} \\
&= \frac{q^2(p+2)}{p(q-2)(q-4)} \\
\text{Var}[F] &= \mathbb{E}[F^2] - (\mathbb{E}[F])^2 \\
&= \frac{q^2(p+2)}{p(q-2)(q-4)} - \frac{q^2}{(q-2)^2} \\
&= \frac{2q^2(p+q-2)}{p(q-4)(q-2)^2}
\end{aligned}$$

□

Theorem 1.12. If $X \sim F_{p,q}$, then $\frac{1}{X} \sim F_{q,p}$.

Proof. By Definition 1.10, $X = \frac{U/p}{V/q}$, where $U \sim \chi^2_p$ and $V \sim \chi^2_q$ are independent.
Then $\frac{1}{X} = \frac{V/q}{U/p} \sim F_{q,p}$ by Definition 1.10 again. □

Theorem 1.13. If $X \sim F_{p,q}$, then $\frac{\frac{p}{q}X}{1+\frac{p}{q}X} \sim \text{Beta}(\frac{p}{2}, \frac{q}{2})$.

Proof. Let $Y = \frac{\frac{p}{q}X}{1+\frac{p}{q}X}$. Hence, $X = \frac{qY}{p(1-Y)} \implies \frac{dx}{dy} = \frac{q}{p}\frac{1-y-(-1)y}{(1-y)^2} = \frac{q}{p(1-y)^2}$. The density of Y is:

$$\begin{aligned}
f_Y(y) &= f_X\left(\frac{qy}{p(1-y)}\right)\left|\frac{dx}{dy}\right| \\
&= \frac{\Gamma(\frac{p+q}{2})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}\left(\frac{p}{q}\right)^{\frac{p}{2}} \frac{\left(\frac{qy}{p(1-y)}\right)^{\frac{p}{2}-1}}{\left(1+\frac{p}{q}\frac{qy}{p(1-y)}\right)^{\frac{p+q}{2}}} \frac{q}{p(1-y)^2} \\
&= \frac{1}{B(\frac{p}{2},\frac{q}{2})}\left(\frac{y}{1-y}\right)^{\frac{p}{2}-1}\left(1+\frac{y}{1-y}\right)^{-\frac{p+q}{2}} \frac{1}{(1-y)^2} \\
&= \frac{1}{B(\frac{p}{2},\frac{q}{2})} y^{\frac{p}{2}-1}(1-y)^{-\frac{p}{2}+1+\frac{p+q}{2}-2} \\
&= \frac{1}{B(\frac{p}{2},\frac{q}{2})} y^{\frac{p}{2}-1}(1-y)^{\frac{q}{2}-1}
\end{aligned}$$

This is the probability density function of $Y \sim \text{Beta}(\frac{p}{2}, \frac{q}{2})$. □

Theorem 1.14 (Properties of t-distribution)**.** Let T be a random variable with a t-distribution of p degrees of freedom ($p \geq 2$ and $p \in \mathbb{N}^*$), i.e. $T \sim t_p$. Then

i. The probability density function of T is

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \cdot \frac{1}{\sqrt{\pi p}} \cdot \frac{1}{\left(\frac{t^2}{p}+1\right)^{\frac{p+1}{2}}}, \quad \forall t \in \mathbb{R}$$

ii. The expectation and variance of T is

$$\mathbb{E}[T] = 0, \ \text{Var}[T] = \begin{cases} \frac{p}{p-2}, & \text{when } p \geq 3 \\ \text{undefined}, & \text{when } p = 2 \end{cases}$$

Proof. i. By Definition 1.9,

$$T = \frac{U}{\sqrt{\frac{V}{p}}}$$

where $U \sim \mathcal{N}(0,1)$ and $V \sim \chi^2_p$ are independent.

The joint probability density function of U and V is

$$\begin{aligned}
f_{UV}(u,v) &= f_U(u)f_V(v) \quad \text{by independence} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}} \frac{1}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}} v^{\frac{p}{2}-1}e^{-\frac{v}{2}}\mathbb{1}_{v>0}, \quad \forall u,v \in \mathbb{R}
\end{aligned}$$

Let $W := V$. We are going to change from variables (U,V) to (T,W). The Jacobian is

$$J = \begin{bmatrix} \frac{\partial T}{\partial U} & \frac{\partial T}{\partial V} \\ \frac{\partial W}{\partial U} & \frac{\partial W}{\partial V} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{p}{V}} & \frac{\partial T}{\partial V} \\ 0 & 1 \end{bmatrix} \implies \det J = \sqrt{\frac{p}{V}}$$

For the realized values $T = t$ and $W = w$,

$$u = \sqrt{\frac{w}{p}}t, v = w$$

Then the joint pdf of T and W is

$$
\begin{aligned}
f_{TW}(t,w) &= (\det J)^{-1} f_{UV}\left(\sqrt{\frac{w}{p}}t, w\right) \\
&= \sqrt{\frac{w}{p}}\frac{1}{\sqrt{2\pi}}e^{-\frac{wt^2}{2p}}\frac{1}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}}w^{\frac{p}{2}-1}e^{-\frac{w}{2}}\mathbb{1}_{w>0}
\end{aligned}
$$

The pdf of T is a marginal distribution and can be calculated as

$$
\begin{aligned}
f_T(t) &= \int_{\mathbb{R}} f_{TW}(t,w)dw \\
&= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{\frac{p}{2}}\sqrt{p}}\int_0^\infty w^{\frac{p+1}{2}-1}e^{-\frac{1}{2}\left(\frac{t^2}{p}+1\right)w}dw \\
&= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\Gamma\left(\frac{p}{2}\right)2^{\frac{p}{2}}\sqrt{p}}\Gamma\left(\frac{p+1}{2}\right)\left(\frac{2}{\frac{t^2}{p}+1}\right)^{\frac{p+1}{2}} \quad \text{let } k=\frac{p+1}{2} \text{ and } \theta=\frac{2}{\frac{t^2}{p}+1} \text{ for a Gamma random variable} \\
&= \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\cdot\frac{1}{\sqrt{\pi p}}\cdot\frac{1}{\left(\frac{t^2}{p}+1\right)^{\frac{p+1}{2}}}
\end{aligned}
$$

ii. The expectation and variance of T can be calculated by merely evaluating integrals:

$$
\mathbb{E}[T] = \int_{-\infty}^{\infty} t\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\cdot\frac{1}{\sqrt{\pi p}}\cdot\frac{1}{\left(\frac{t^2}{p}+1\right)^{\frac{p+1}{2}}}dt
$$

Note that the integrand is an odd function, so the integral equals 0, i.e., $\mathbb{E}[T]=0$. For the variance:

$$
\begin{aligned}
\text{Var}[T] &= \mathbb{E}[T^2] - (\mathbb{E}[T])^2 \\
&= \int_{-\infty}^{\infty} t^2\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\cdot\frac{1}{\sqrt{\pi p}}\cdot\frac{1}{\left(\frac{t^2}{p}+1\right)^{\frac{p+1}{2}}}dt - 0 \\
&= 2\int_0^{\infty} t^2\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\cdot\frac{1}{\sqrt{\pi p}}\cdot\frac{1}{\left(\frac{t^2}{p}+1\right)^{\frac{p+1}{2}}}dt \quad \text{the integrand is an even function} \\
&= 2\int_0^{\infty} u\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\cdot\frac{1}{\sqrt{\pi p}}\cdot\frac{1}{\left(\frac{u}{p}+1\right)^{\frac{p+1}{2}}}\cdot\frac{du}{2\sqrt{u}} \quad \text{let } u = x^2 \\
&= \int_0^{\infty} u\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(\frac{1}{p}\right)^{\frac{1}{2}}u^{\frac{1}{2}-1}\left(1+\frac{u}{p}\right)^{-\frac{p+1}{2}}du \quad \text{using } \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}
\end{aligned}
$$

By Theorem 1.11, the integrand is that of u multiplied by the pdf of an F distribution with 1 and p degrees of freedom, which is equal to $\frac{p}{p-2}$. So $\text{Var}[T]=\mathbb{E}[T^2]=\frac{p}{p-2}$ for $p>2$. When $p=2$, the pdf of T is the same as that of a standard Cauchy distribution, where the variance is undefined.

□

Theorem 1.15 (Connection of t-distribution and Normal distribution)**.** Let $T\sim t_p$. Then $T \rightsquigarrow \mathcal{N}(0,1)$ as $p\to\infty$

Proof. The density of T is

$$
\begin{aligned}
f_T(t) &= \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}\cdot\frac{1}{\sqrt{\pi p}}\cdot\frac{1}{\left(\frac{t^2}{p}+1\right)^{\frac{p+1}{2}}} \\
&= \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\sqrt{\pi p}}\left(1-\frac{t^2}{t^2+p}\right)^{\frac{p+1}{2}}
\end{aligned}
$$

$$\lim_{p\to\infty} f_T(t) = \lim_{p\to\infty} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})\sqrt{\pi p}} \lim_{p\to\infty} \left(1 - \frac{t^2}{t^2+p}\right)^{\frac{p+1}{2}}$$

For the first term, we rely on Stirling's approximation for Gamma functions:

$$\lim_{x\to+\infty} \frac{\Gamma(x)}{\sqrt{2\pi(x-1)}\left(\frac{x-1}{e}\right)^{x-1}} = 1$$

$$\begin{aligned}
\lim_{p\to\infty} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})\sqrt{\pi p}} &= \lim_{p\to\infty} \frac{\sqrt{2\pi\frac{p-1}{2}}\left(\frac{p-1}{2}\right)^{\frac{p-1}{2}} e^{-\frac{p-1}{2}}}{\sqrt{\pi p}\sqrt{2\pi\frac{p-2}{2}}\left(\frac{p-2}{2}\right)^{\frac{p-2}{2}} e^{-\frac{p-2}{2}}} \\
&= \lim_{p\to\infty} \sqrt{\frac{p-1}{\pi p(p-2)}} e^{-\frac{1}{2}} \left(\frac{p-1}{2}\right)^{\frac{1}{2}} \left(\frac{p-1}{p-2}\right)^{\frac{p-2}{2}} \\
&= \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \lim_{p\to\infty} \sqrt{\frac{(p-1)^2}{p(p-2)}} \lim_{p\to\infty} \left(1 + \frac{1}{p-2}\right)^{\frac{p-2}{2}} \\
&= \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \cdot 1 \cdot \sqrt{e} = \frac{1}{\sqrt{2\pi}}
\end{aligned}$$

The second term is nonnegative, so

$$\lim_{p\to\infty} \left(1 - \frac{t^2}{t^2+p}\right)^{\frac{p+1}{2}} = \left(\lim_{p\to\infty} \left(1 - \frac{t^2}{t^2+p}\right)^{p+1}\right)^{\frac{1}{2}} = \sqrt{e^{-t^2}} = e^{-\frac{t^2}{2}}$$

Combining the two terms,

$$\lim_{p\to\infty} f_T(t) = \lim_{p\to\infty} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})\sqrt{\pi p}} \lim_{p\to\infty} \left(1 - \frac{t^2}{t^2+p}\right)^{\frac{p+1}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

which is the pdf of standard normal distribution. □

Theorem 1.16 (Connection of t and χ^2 distributions)**.** If $X \sim t_p$, then $X^2 \rightsquigarrow \chi_1^2$ as $p \to \infty$

Proof. As $p \to \infty$, $X \rightsquigarrow \mathcal{N}(0,1)$ by Theorem 1.15. Since the mapping $x \mapsto x^2$ is continuous, by Theorem 1.7 and Theorem 5.5 (g) in *All of Statistics (2004)* by Wasserman [2], $t_p \rightsquigarrow \chi_1^2$. □

Theorem 1.17 (Connection of t and F distributions)**.** If $X \sim t_p$, then $X^2 \sim F_{1,p}$.

Proof. The density of $Y := X^2$ is

$$\begin{aligned}
f_Y(y) &= (f_X(\sqrt{y}) + f_X(-\sqrt{y})) \left|\frac{dx}{dy}\right| \\
&= 2 f_X(\sqrt{y}) \left|\frac{dx}{dy}\right| \quad \text{as } f_X \text{ is an even function} \\
&= 2 \left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi p}\,\Gamma(\frac{p}{2})} \frac{1}{\left(\frac{y}{p}+1\right)^{\frac{p+1}{2}}}\right) \frac{1}{2\sqrt{y}} \quad \text{by Theorem 1.14} \\
&= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{p}{2})} \left(\frac{1}{p}\right)^{\frac{1}{2}} \frac{y^{\frac{1}{2}-1}}{\left(1 + \frac{1}{p}y\right)^{\frac{p+1}{2}}}
\end{aligned}$$

which is the density of $Y \sim F_{1,p}$ by Theorem 1.11. □

References

[1] G. Casella and R.L. Berger. *Statistical Inference*. Cengage Learning, 2021.

[2] Larry Wasserman. *All of statistics: a concise course in statistical inference*, volume 26. Springer, 2004.