1 Motivation

We recall that an element $X_p \in T_p(M)$ can be viewed as a function $C^\infty(M) \to \mathbb{R}$ which takes $f \to$ the directional derivative of f at p in the direction X_p.

This extends naturally to vector fields: a vector field X on M can be viewed as an operator $C^\infty(M) \to C^\infty(M)$ which takes a function f to the function Xf, where the argument of Xf is a point $p \in M$ and the output is $X_p f$.

Could we extend this idea further by having a vector field X able to act on arbitrary tensor fields, rather than just $(0,0)$ tensor fields (which are just functions)? We will use the notation ∇_X to describe the desired operator taking one (p,q) tensor field to another according to the action of X.

2 (Affine) Connections

Definition 2.1. A **connection** is a map $\nabla : \mathscr{X}(M) \times T^p_q(M) \to T^p_q(M)$ which takes a pair (X,T) consisting of a vector field and (p,q)-tensor field to another (p,q)-tensor field $\nabla_X T$ satisfying the following properties:

(i) For $f \in C^\infty(M), \nabla_X f = Xf$.

(ii) $\nabla_X(T+S) = \nabla_X(T) + \nabla_X(S)$

(iii) (Leibnitz Rule) $\nabla_X(fY) = f\nabla_X(Y) + (\nabla_X f)Y$

(iv) $\nabla_{fX+gZ}T = f\nabla_X T + g\nabla_Z T$.

In order to make ∇ uniquely defined, we need to add some additional structure to our manifold.

(Expanded (iii): Leibnitiz Rule is basically product rule for Tensor Products:

$$\nabla_X(T \otimes S) = \nabla_X T \otimes S + T \otimes \nabla_X S$$

If T is a (p,q)-tensor, $\omega_1, \dots, \omega_p, Y_1, \dots, Y_q$ covectors and vectors respectively, then

$$\begin{gathered}\nabla_X(T(\omega_1, \dots, \omega_p, Y_1, \dots, Y_q)) \\ = (\nabla_X T)(\omega_1, \dots, \omega_p, Y_1, \dots, Y_q) \\ +T(\nabla_X \omega_1, \omega_2, \dots, \omega_p, Y_1, \dots, Y_q) + \cdots + T(\omega_1, \omega_2, \dots, \nabla_X \omega_p, Y_1, \dots, Y_q) \\ +T(\omega_1, \omega_2, \dots, \omega_p, \nabla_X Y_1, \dots, Y_q) + \cdots + T(\omega_1, \omega_2, \dots, \omega_p, Y_1, \dots, \nabla_X Y_q)\end{gathered}$$

3 Connection Coefficient Functions

Note that for a given chart (U,x),

$$\begin{aligned}\nabla_X Y &= \nabla_{X^k \frac{\partial}{\partial x^k}} Y^j \frac{\partial}{\partial x^j} \\ &= X^k \nabla_{\frac{\partial}{\partial x^k}} Y^j \frac{\partial}{\partial x^j} && \text{(by linearity)} \\ &= X^k (\nabla_{\frac{\partial}{\partial x^k}} Y^j) \frac{\partial}{\partial x^j} + X^k Y^j (\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}) && \text{(by Leibnitz rule)} \\ &= X^k (\frac{\partial}{\partial x^k} Y^j) \frac{\partial}{\partial x^j} + X^k Y^j \Gamma^l_{kj} \frac{\partial}{\partial x^l}\end{aligned}$$

where the last step follows first from the fact that the Y^j are just functions, second term since $\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}$ is a vector field thus we can expand it under the chart, the Γ coefficients are summed and also depend on k and j hence the subscript). Note that the Γ^i_{kj} are dependent on our choice of chart.

We have defined the Γs implicitly via the equation $\Gamma^l_{jk} \frac{\partial}{\partial x^l} = (\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j})$. For a given i, applying the covector dx^i to both sides gives us $dx^i(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}) = dx^i \Gamma^l_{jk} \frac{\partial}{\partial x^l} = \Gamma^l_{jk} dx^i \frac{\partial}{\partial x^l} = \Gamma^l_{jk} \delta^i_m = \Gamma^i_{jk}$. This leads us to the following definition:

Definition 3.1. Given a manifold M with connection and a chart (U, x), then the **connection coefficient functions** are the $(\dim(M))^3$ functions

$$\Gamma^i_{jk} : U \to \mathbb{R}, \quad p \mapsto dx^i(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j})(p)$$

From our earlier equation $\nabla_X Y = X^k(\frac{\partial}{\partial x^k} Y^l + Y^j \Gamma^l_{jk}) \frac{\partial}{\partial x^l}$, we see that the domain of a chart U, the choice of these $(\dim(M))^3$ functions Γ^i_{jk} is sufficient to fix the action ∇_X (given an vector field X) on any other vector field Y.

4 Change of Γs under change of chart

Given charts (U, x) and (V, y) with nonempty intersection, lets try to relate the Γ wrt the y chart to the Γ wrt the x chart.

$$\Gamma^l_{(y)jk} \frac{\partial}{\partial x^l} = \nabla_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j}$$

Then, just using change of variables for every occurrence of y, we get that the above equals:

$$\left(\nabla_{\frac{\partial x^\alpha}{\partial y^k}\frac{\partial}{\partial x^\alpha}} \frac{\partial x^s}{\partial y^j} \frac{\partial}{\partial x^s}\right)$$

$$= \frac{\partial x^\alpha}{\partial y^k} \nabla_{\frac{\partial}{\partial x^\alpha}} \left(\frac{\partial x^s}{\partial y^j} \frac{\partial}{\partial x^s}\right) \text{ (Linearity axiom)}$$

$$= \frac{\partial x^\alpha}{\partial y^k} \left(\left(\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial x^s}{\partial y^j}\right) \frac{\partial}{\partial x^s} + \left(\frac{\partial x^s}{\partial y^j} \nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^s}\right)\right) \text{ (Leibnitz)}$$

$$= \left(\frac{\partial x^\alpha}{\partial y^k} \left(\frac{\partial}{\partial x^\alpha} \frac{\partial x^s}{\partial y^j}\right) \frac{\partial}{\partial x^s}\right) + \left(\frac{\partial x^\alpha}{\partial y^k} \left(\frac{\partial x^s}{\partial y^j} \Gamma^l_{(x)\alpha s} \frac{\partial}{\partial x^l}\right)\right) \text{ (Distribution \& replacing parenthesized terms)}$$

$$= \frac{\partial^2 x^s}{\partial y^k \partial y^j} \frac{\partial}{\partial x^s} + \left(\frac{\partial x^\alpha}{\partial y^k} \frac{\partial x^s}{\partial y^j}\right) \Gamma^l_{(x)\alpha s}$$

Hence, we see that if the terms $\frac{\partial^2 x^s}{\partial y^k \partial y^j}$ are nonzero, it is entirely possible for the Γ for y to be all zero while the Γs for x are not. Under a linear change of coordinates (with second-derivatives zero), we know vanishing Γs are taken to vanishing Γs.

From the above, given choices of Γs on overlapping charts U, V, the above relation must be satisfied. Hence we can read it as a compatibility condition, and we see that we cannot freely choose the Γs for the charts across the manifold without paying attention to the above identity.

5 Geodesics

Connections are very helpful in giving a characterization of Geodesics that's different from the one given previously in our course, which characterized it as the shortest possible path between two points on a surface. In this section, we will introduce an alternative defition that allude to its connection with parallelisms.

Definition 5.1 (Connection Restricted to a Curve)**.** Let $\gamma : (a, b) \to M$ be a curve, then locally wrt chart (U, x) we can paramaterize each component of γ as

$$\gamma^i(t) = x^i \circ \gamma(t)$$

In particular, we note that as a curve $\gamma(t)$, its directional gives a natural vector field defined on $\gamma(t)$, locally we have that

$$\gamma'(t) = \frac{d\gamma^i(t)}{dt} \frac{\partial}{\partial x^i}|_\gamma$$

Now let Y be an arbitrary vector field also defined on γ, then we define

$$\nabla_{\gamma'} Y := \nabla_X \tilde{Y}|_\gamma = (\frac{dY^i}{dt} + \Gamma^i_{jk} Y^k \frac{d\gamma^j}{dt}) \frac{\partial}{\partial x^i}|_\gamma$$

where $X, \tilde{Y}$ are global extensions of γ' and Y respectively. The key idea here is that connection on a curve is the same as restricting a bigger vector field down to that curve.

Definition 5.2 (Parallel Lines)**.** Let Y be a vector field defined on the curve $\gamma : (a, b) \to M$, if $\nabla_{\gamma'} Y = 0$, then we say that Y **is parallel along the curve** γ.

Definition 5.3 (Geodesics)**.** When $Y = \gamma'$, so when γ' is parallel along the curve γ ($\nabla_{\gamma'}\gamma' = 0$), we say that γ is a **geodesic** on M.

Remark 5.4. Intuitively, a geodesic is a curve for which tangent vectors parallel to the curve remain parallel upon transportation along the curve.

Remark 5.5. Using local coordinates on M, we can write $\nabla_{\gamma'} Y = 0$ as

$$\forall i, \frac{dY^i}{dt} + \Gamma^i_{jk} Y^k \frac{d\gamma^j}{dt} = 0$$

When $Y = \gamma'$, the differential equation above becomes

$$\forall i, \frac{d^2\gamma^i}{dt^2} + \Gamma^i_{kj} \frac{d\gamma^k}{dt}\frac{d\gamma^j}{dt} = 0$$

This is a necessary and sufficient condition for γ to be a geodesic on M.

Using some knowledge from ODEs, the equation above has unique solutions with a given an initial position and an initial velocity. Thus, geodesics can be used to visualize trajectories of free particles in a manifold where its motion is completely determined by the bending of the manifold.

For general relativity enthusiasts, this is what it means for particles to move on the surface and the bending of the manifold is caused by gravity.

6 Riemannian Connection

Previously, we have defined connections on an arbitrary differentiable manifold. However, in differential geometry, we are more interested in connections on a Riemannian manifold.

Definition 6.1. Let (M, g) be a smooth Riemannian manifold, a connection ∇ on M is called a **Riemannian connection** with respect to g if for all tangent vectors X, Y, Z, it satisfies the following two conditions:

- (a - "Symmetry") $\nabla_X Y - \nabla_Y X = [X, Y]$
- (b - "Metric Compatibility") $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, where $g(.,.)$ is the inner product for g

Remark 6.2. Riemannian connection is also called **Levi-Civitas connection**. Property (a) indicates that Riemannian connections are symmetrcial; Property (b) is the same as saying that the inner product given by g of any two ∇-parallel vector fields along a given curve is constant.
Thus, a Riemannian connection is a symmetrical connection that preserves the Riemannian metric.

Remark 6.3. Recall from our axioms for ∇ that $X(g(Y, Z)) = \nabla_X(g(Y, Z))$. By the generalized form of the Leibnitz rule, this equals $(\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. Hence we see that axiom (b) is equivalent to
$\nabla_X(g)(Y, Z) = 0 \iff \nabla_X g = 0$, i.e. we require that all covariant derivatives of our metric tensor vanish.

Theorem 6.4 (The Fundamental Theorem of Riemannian Geometry)**.** Let (M, g) be a C^2 Riemannnian manifold, then there exists a unique Riemmanian connection ∇ on (M, g)

Proof. **1. Uniqueness:** It suffcies for us to prove that $g(\nabla_X Y, Z)$ is uniquely determined, this is because for any two Riemannian connections ∇^1, ∇^2, and Z non-zero,

$$g(\nabla^1_X Y, Z) = g(\nabla^2_X Y, Z) \implies 0 = g(\nabla^1_X Y - \nabla^2_X Y, Z) \implies 0 = \nabla^1_X Y - \nabla^2_X Y$$
$$\implies \nabla^1_X Y = \nabla^2_X Y$$

Since ∇^1, ∇^2 agree on all the tangent vector fields, they have to be the same operator.

Now from Property (b), we have that

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$-Zg(X,Y) = -g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

If we add the three equations together, we have that

$$\begin{aligned}
Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\
&= g(\nabla_X Y, Z) + g(\nabla_X Z, Y) + g(\nabla_Y Z, X) + g(\nabla_Y X, Z) - g(\nabla_Z X, Y) - g(\nabla_Z Y, X) \\
&\qquad \text{g is symmetric} \\
&= g(\nabla_X Y + \nabla_Y X, Z) + \mathbf{g}(\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{X}, \mathbf{Y}) + \mathbf{g}(\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{Y}, \mathbf{X}) \qquad \text{Bilinearity} \\
&= g(\nabla_X Y + \nabla_Y X, Z) + g([X,Z], Y) + g([Y,Z], X) \qquad \text{Property (a)} \\
&= g(\nabla_X Y \underline{+\nabla_X Y - \nabla_X Y} + \nabla_Y X, Z) + g([X,Z], Y) + g([Y,Z], X) \\
&= g(2\nabla_X Y, Z) + g(\nabla_Y X - \nabla_X Y, Z) + g([X,Z], Y) + g([Y,Z], X) \\
&= 2g(\nabla_X Y, Z) + g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X)
\end{aligned}$$

So we have that

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X)] \ (*)$$

This is what's known as the **Koszul formula**, and we see that $g(\nabla_X Y, Z)$ is completely independent of the choice of ∇, so it's uniquely determined.

2. Existence: We can use the formula $(*)$ above to define a connection $\nabla_X Y$, then one can verify that this said ∇ satisfied both Property (a) and (b), the verification is left to you, the enthusiastic audience. Thus, existence is proven. □

Remark 6.5. A manifold with a metric tensor that is nondegenerate but not necessarily positive-definite is called a **pseudo-Riemannian** manifold. Note that throughout this proof we never used the positive definiteness of g, only its nondegeneracy. Hence, this theorem still holds for pseudo-Riemannian manifolds. One important example is the 4-dimensional Lorentzian manifold, which is used for modelling spacetime.

Remark 6.6. Using the Koszul formula given above in a local coordinate $\{x^i\}$, take

$$X = \frac{\partial}{\partial x^j}, Y = \frac{\partial}{\partial x^k}, Z = \frac{\partial}{\partial x^\ell}$$

, and recall that

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}$$

On the LHS,

$$\begin{aligned}
2g(\nabla_X Y, Z) &= 2g(\Gamma^i_{jk} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\ell}) \\
&= 2\Gamma^i_{jk} g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^\ell}) \\
&= 2\Gamma^i_{jk} g_{i\ell}
\end{aligned}$$

On the RHS, the last 3 terms disappears immediately when substituted since $[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}] = 0$, so we have that

$$RHS = \frac{\partial}{\partial x^j} g_{k\ell} + \frac{\partial}{\partial x^k} g_{j\ell} - \frac{\partial}{\partial x^\ell} g_{jk}$$

In particular, we can solve Γ^i_{jk} as:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} (\frac{\partial g_{k\ell}}{\partial x^j} + \frac{\partial g_{j\ell}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell})$$

, where $(g^{k\ell})$ is the inverse of the matrix system $(g_{\ell s})$ defined as $g^{k\ell} g_{\ell s} = \delta^k_s$.

Therefore, we can calculate exactly what the Riemannian connection of a given (M, g) based on g itself.

Γ^i_{jk} is called the **Christoffel symbols of the second kind** with respect to the Riemannian metric g.

Definition 6.7. Let ∇ be the Riemannian connection on (M, g), let

$$Q(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

We note that $Q(X,Y) : \mathscr{X}(M) \to \mathscr{X}(M)$ is an endomorphism on vector fields. Then, we defined $R(X,Y,Z,W)$ as

$$R(X,Y,Z,W) := g(Q(Z,W)Y, X)$$

This is called the **Riemann curvature tensor**.

Remark 6.8. We want a more explicit form of what R is. Since R is a $(0,4)$-tensor, it suffices for us to examine how R acts on the basis $\{\frac{\partial}{\partial x^i}\}$ of the tangent vector space.

With some ease of notation, we will denote $e_i = \frac{\partial}{\partial x^i}$ because (we will be writing this a lot of times).

We are interested in the term

$$R_{ijk\ell} = R(e_i, e_j, e_k, e_\ell) = g(Q(e_k, e_\ell)e_j, e_i)$$

We look at the term $Q(e_k, e_\ell)e_j$:

$$\begin{aligned} Q(e_k, e_\ell)e_j &= \nabla_{e_k}\nabla_{e_\ell}e_j - \nabla_{e_\ell}\nabla_{e_k}e_j && \text{Lie Bracket vanishes} \\ &= \nabla_{e_k}\Gamma^\alpha_{j\ell}e_\alpha - \nabla_{e_\ell}\Gamma^\alpha_{jk}e_\alpha \\ &= (\nabla_{e_k}\Gamma^\alpha_{j\ell})e_\alpha + \Gamma^\alpha_{j\ell}(\nabla_{e_\ell}e_\alpha) - (\nabla_{e_\ell}\Gamma^\alpha_{jk})e_\alpha - \Gamma^\alpha_{jk}(\nabla_{e_\ell}e_\alpha) \\ &= (e_k\Gamma^\alpha_{j\ell} + \Gamma^\alpha_{j\ell}\Gamma^\beta_{\alpha\ell}e_\beta - e_\ell\Gamma^\alpha_{jk} - \Gamma^\alpha_{jk}\Gamma^\beta_{\alpha\ell}e_\beta)e_\alpha \\ &= S^\alpha_{jk\ell}e_\alpha \end{aligned}$$

Thus, we have

$$\begin{aligned} R_{ijk\ell} &= g(S^\alpha_{jk\ell}e_\alpha, e_i) \\ &= S^\alpha_{jk\ell}\ g(e_\alpha, e_i) && \text{Bilinearity} \\ &= S^\alpha_{jk\ell}\ g_{\alpha i} && \text{Definition of Riemannian Metric} \end{aligned}$$

Thus, we have that

$$R_{ijk\ell} = S^\alpha_{jk\ell}g_{\alpha i}$$

Everything after this won't be covered in the lecture, but we thought it's interesting to attach it here.

Proposition 6.9. Let R be a Riemann curvature tensor, then

- (1) $R_{\underline{ij}k\ell} = -R_{\underline{ji}k\ell}$
- (2) $R_{ij\underline{k\ell}} = R_{ij\underline{\ell k}}$
- (3) $R_{i\underline{jk\ell}} + R_{i\underline{k\ell j}} + R_{i\underline{\ell jk}} = 0$ (Sum of permuting the last 3 index is 0)
- (4) $R_{ijk\ell} = R_{k\ell ij}$

Proof. Exercise □

Remark 6.10. Property (3) is what's known as the **First Bianchi Identity**, don't ask what the second is. Intuitively, the idea is that the three curvature tensors listed should form a triangle, meaning that their sum is 0.

Property $(1)-(3)$ is a complete characterization of the Riemann curvature tensor. To be more precise, for any $(0,4)$-tensor T that satisfies Property (1) to (3), one can construct a Riemann manifold (M, g, ∇) with T being its curvature tensor.

Corollary 6.11. Let (M, g, ∇) be an m-dimensional Riemannian manifold, its Riemann curvature tensor R has $\frac{m^2(m^2-1)}{12}$ algebraically independent components.

Proof. This is a fun combinatorics exericse. Try to play around with Property (1) to (3). □

Remark 6.12. When M is a 2-dimensional Riemannian manifold, then its curvature tensor R has only 1 algebraically independent component. In other words, it's enough to characterize M with a scalar value. $\frac{R}{2}$ coincides with the **Gaussian Curvature** we learned in Calculus III.